THE CONSTRUCTION OF MAGIC SQUARES AND RECTANGLES BY THE METHOD OF "COMPLEMENTARY DIFFERENCES."[1]

WE are indebted to Dr. C. Planck for a new and powerful method for producing magic squares, rectangles etc. This method is especially attractive and valuable in furnishing a *general* or *universal* rule covering the construction of all conceivable types of squares and rectangles, both odd and even. It is not indeed the easiest and best method for making *all kinds* of squares, as in many cases much simpler rules can be used to advantage, but it will be found exceedingly helpful in the production of new variants, which might otherwise remain undiscovered, seeing that they may be non-La Hireian and ungoverned by any obvious constructive plan.

When a series of numbers is arranged in two associated columns, as shown in Fig. 1, each pair of numbers has its distinctive difference, and these "complementary differences," as they are termed by Dr. Planck, may be used very effectively in the construction of magic squares and rectangles. In practice it is often quite as efficient and simpler to use the half differences, as given in Fig. 1.

In illustrating this method we will first apply it to the construction of an associated or regular 3×5 magic rect-

[1] This article has been compiled almost entirely from correspondence received by the writer from Dr. Planck, and in a large part of it the text of his letters has been copied almost verbatim. Its publication in present form has naturally received his sanction and endorsement. W. S. A.

angle, in which the natural numbers 1 to 15 inclusive are to be so arranged that every long row sums 40, and every short column sums 24. The center cell must necessarily be occupied by 8, which is the middle number of the series, and the complementary numbers must lie in associated cells, such as *a a* — *b b* — *c c* in Fig. 2.

The first operation is to lay out a 3×5 rectangle and fill it with such numbers that all the short columns shall sum 24, but in which the numbers in the columns will not be placed in any particular order. When two columns of this rectangle are filled, three pairs of complementary

1	15	7
2	14	6
3	13	5
4	12	4
5	11	3
6	10	2
7	9	1
8		

Fig. 1.

a				b
	c		c	
b				a

Fig. 2.

2	5	4	3	1
7	6	8	10	9
15	13	12	11	14

Fig. 3.

numbers will have been used, and their differences will have disappeared, as these two columns must each sum 24. Hence, *one complementary difference must equal the sum of the other two.*

We have therefore (neglecting the middle column) to make two equations of the form $a = b + c$ from the complementary differences, without using the same difference twice. Thus:

$$\left.\begin{array}{l} 7 = 6 + 1 \\ 5 = 3 + 2 \end{array}\right\} \dots\dots\dots\dots\dots\dots \text{(I)}$$

is such a pair of equations.

The first equation indicates that the greater of the two complements whose half difference is 7 can lie in the same column with the lesser members of the pairs whose half differences are 6 and 1. In other words, the numbers 15, 7 and 2 can lie in one column, and their complements 14, 9 and 1 in the associated column. The second equation $(5 = 3 + 2)$ gives similar information regarding the other pair of associated columns, and the three remaining numbers must then be placed in the middle column, thus producing the rectangle shown in Fig. 3.

These equations determine nothing as to the placing of the numbers in the rows, since in Fig. 3 the numbers in the columns have no definite order.

The rows may now be attacked in a similar manner. Two of the complementary differences in the upper or lower row must equal the other three, and the equation will therefore be of the order $a + b = c + d + e$.

In order that the disposition of numbers in the columns shall not be disturbed, the numbers used in this equation must be so chosen that any *two* numbers which appear together on the *same side* of an equality sign in the short column equation, must not so appear in a long row equation, also if two numbers appear on the *opposite sides* of an equality sign in a short column equation, *they must not so appear* in the long row equation.

There is only one such equation which will conform to the above rules, viz.:

$$6 + 2 = 4 + 3 + 1.$$

Interpreting this as before we have the rectangle given in Fig. 4, in which each of the three rows sums 40. We have now two rectangles, Fig. 3 showing the correct numbers in the columns, and Fig. 4 showing the proper disposition of the numbers in the rows. By combining them

we get the associated or regular magic rectangle given in Fig. 5.

If a mere shuffling of pairs of complementary rows or columns is ignored, this is the *only* solution of the problem.[2]

4	5	7	10	14
1	3	8	13	15
2	6	9	11	12

Fig. 4.

7	5	4	10	14
15	13	8	3	1
2	6	12	11	9

Fig. 5.

There are two pairs of equations of the form

$$a = b + c$$
$$d = e + f$$

namely, the one given in (I) and

$$\left.\begin{array}{l} 7 = 5 + 2 \\ 4 = 3 + 1 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(II)}$$

and there are nine equations of the form

$$a + b = c + d + e$$

but of these nine equations only one will go with (I) and none will go with (II) so as to conform with the above rules.

If the condition of association is relaxed there are thirty-nine different 3×5 magic rectangles.

This method can naturally be used for constructing all sizes of magic rectangles which are possible,[3] but we will only consider one of 5×7 as a final example.

[2] The solution of this problem of the associated rectangle is the first step in the construction of the higher ornate magics of composite odd orders. For example, if the above single solution for the 3×5 rectangle did not exist it would be impossible to construct a magic, pan-diagonal, associated (= regular) square of order 15, which shall be both 9-ply and 25-ply, i. e., *any* square bunch of 9 cells to sum up 9 times the mean, and *any* square bunch of 25 cells 25 times the mean. C. P.

[3] A magic rectangle with an odd number of cells in one side and an even number in the other, is impossible with consecutive numbers. C. P.

Fig. 6 shows the associated series of natural numbers from 1 to 35 with their half differences, from which the numbers must be chosen in accordance with the above

1	35	17
2	34	16
3	33	15
4	32	14
5	31	13
6	30	12
7	29	11
8	28	10
9	27	9
10	26	8
11	25	7
12	24	6
13	23	5
14	22	4
15	21	3
16	20	2
17	19	1
18		

Fig. 6.

19	22	33	29	23	21	20
35	31	34	28	30	24	25
9	10	4	18	32	26	27
11	12	6	7	2	5	1
16	15	13	8	3	14	17

Fig. 7.

30	31	34	1	7	9	14
25	26	28	16	15	13	3
32	24	19	18	17	12	4
33	23	21	20	8	10	11
22	27	29	35	2	5	6

Fig. 8.

9	31	34	7	30	14	1
16	15	13	28	3	26	25
19	12	4	18	32	24	17
11	10	33	8	23	21	20
35	22	6	29	2	5	27

Fig. 9.

rules. In this case there will be three equations of the order

$$a + b = c + d + e$$

for the columns, and two equations of the order

$$a + b + c = d + e + f + g$$

for the rows. The following selection of numbers will satisfy the conditions:

$$\left.\begin{aligned} 1 + 17 &= 9 + 7 + 2 \\ 4 + 13 &= 8 + 6 + 3 \\ 15 + 16 &= 14 + 12 + 5 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \text{(III)}$$

for the columns, and

$$\left.\begin{aligned} 12 + 13 + 16 &= 17 + 11 + 9 + 4 \\ 7 + 8 + 10 &= 2 + 3 + 5 + 15 \end{aligned}\right\} \quad \ldots\ldots \quad \text{(IV)}$$

for the rows.

Fig. 7 is a rectangle made from (III) in which all the columns sum 90, and Fig. 8 is a rectangle made from (IV) in which all the rows sum 126. Combining these two rectangles produces Fig. 9 which is magic and associated.

We will now consider this method in connection with magic squares and will apply it to the construction of a square of order 5 as a first example. In this case two equations of the order

$$a + b = c + d + e$$

will be required for the rows and two more similar equations for the columns.

The following will be found suitable for the rows:

$$\left.\begin{aligned} 12 + 11 &= 10 + 9 + 4 \\ 8 + 6 &= 7 + 5 + 2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \text{(V)}$$

and

$$\left.\begin{aligned} 11 + 8 &= 12 + 6 + 1 \\ 10 + 7 &= 9 + 5 + 3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad \text{(VI)}$$

for the columns.

It will be seen that the rule for pairs of numbers in the same equation is fulfilled in the above selection. In (V) 12 and 11 are on the same side of an equality sign, but in (VI) these numbers are on opposite sides, also, 10 and 9 are on the same side in (V) and on opposite sides in (VI) and so on:

The resulting magic square is given in Fig. 10, it is non-La Hireian, and could not easily be made in any way other than as above described.

24	3	9	4	25
21	6	11	8	19
12	16	13	10	14
7	18	15	20	5
1	22	17	23	2

Fig. 10.

The construction of a square of order 6 under this method presents more difficulties than previous examples, on account of the inherent disabilities natural to this square and we will consider it as a final example. The method to be employed is precisely the same as that previously discussed.

For the columns three equations should be made of the form:

$$a+b+c=d+e+f$$

or

$$a+b \quad =c+d+e+f$$

and three similar equations are required for the rows, all being subject to the rule for "pairs and equality sign" as above described. On trial, however, this will be found to be impossible,[4] but if for one of the row- or column-equations we substitute an *inequality* whose difference is 2 we

[4] It is demonstrably impossible for all orders $=4n+2$, i. e., 6, 10, 14 etc. C.P.

shall obtain a square of 6, which will be "associated," but in which two lines or columns will be erratic, one showing a correct summation —1 and the other a correct summation +1. The following equations (VII) may be used for the columns:

$$\left.\begin{aligned} 11+7&=9+5+3+1 \\ 25+17+13&=21+19+15 \\ 35+31+23&=33+29+27 \end{aligned}\right\} \quad .. \text{ (VII)}$$

and for the rows:

$$\left.\begin{aligned} 29+25&=33+13+7+1 \\ 35+19+3&=31+21+15 \\ 27+23&\neq 17+15+11+9 \end{aligned}\right\} \quad .. \text{ (VIII)}$$

the last being an *inequality*. Fig. 11 shows the complementary pairs of natural numbers 1 to 36 with their whole differences, which in this case are used in the equations (VII) and (VIII) instead of the half differences, because these differences can not be halved without involving fractions. Fig. 12 is the square derived from equations (VII) and will be found correct in the columns. Fig. 13 is the square formed from equations (VIII) and is correct in the 1st, 2d, 5th, and 6th rows, but erratic in the 3d and 4th rows. The finished six-square made by combining Figs. 12 and 13 is shown in Fig. 14 which is associated or regular, and which gives correct summations in all the columns and rows excepting the 3d and 4th rows which show —1 and +1 inequalities respectively.

Fig. 14, like Fig. 10, could not probably be produced by any other method than the one herein employed, and both of these squares therefore demonstrate the value of

the methods for constructing new variants. Fig. 14 can be readily converted into a continuous or pan-diagonal square by first interchanging the 4th and 6th columns and then, in the square so formed, interchanging the 4th and

1	36	35
2	35	33
3	34	31
4	33	29
5	32	27
6	31	25
7	30	23
8	29	21
9	28	19
10	27	17
11	26	15
12	25	13
13	24	11
14	23	9
15	22	7
16	21	5
17	20	3
18	19	1

Fig. 11.

24	31	36	35	29	21
22	27	34	33	28	20
14	25	30	32	26	19
16	8	2	1	6	13
17	9	4	3	10	15
18	11	5	7	12	23

Fig. 12.

33	31	2	12	15	18
36	28	20	3	8	16
32	30	10	11	13	14
26	24	23	5	7	27
34	29	21	1	9	17
25	22	19	4	6	35

Fig. 13.

18	31	2	33	12	15
16	8	36	3	28	20
14	11	30	32	10	13
24	27	5	7	26	23
17	9	34	1	29	21
22	25	4	35	6	19

Fig. 14.

and 6th rows. The result of these changes is given in Fig. 15 which shows correct summations in all columns and rows, excepting in the 3d and 6th row which carry

the inequalities shown in Fig. 14. This square has lost its property of association by the above change but has now correct summation in all its diagonals. It is a demonstrable fact that squares of orders $4n+2$, (i. e., 6, 10, 14 etc.) cannot be made perfectly magic in columns and rows and at the same time *either* associated or pandiagonal when constructed with consecutive numbers.

18	31	2	15	12	33
16	8	36	20	28	3
14	11	30	13	10	32
22	25	4	19	6	35
17	9	34	21	29	1
24	27	5	23	26	7

Fig. 15.

A	B
C	D

Fig. 16.

Dr. Planck also points out that the change which converts all even associated squares into pan-diagonal squares may be tersely expressed as follows:

Divide the square into four quarters as shown in Fig. 16.
Leave A untouched.
Reflect B.
Invert C.
Reflect and invert D.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Fig. 17.

7	12	14	1
2	13	11	8
9	6	4	15
16	3	5	10

Fig. 18.

The inverse change from pan-diagonal to association is not necessarily effective, but it may be demonstrated with the "Jaina" square given by Dr. Carus in *Magic*

Squares and Cubes, p. 125, which is here reproduced in Fig. 17. This is a continuous or pan-diagonal square, but after making the above mentioned changes it becomes an associated or regular square as shown in Fig. 18.

1	44	32	53	2	43	31	54
58	19	39	14	57	20	40	13
38	15	59	18	37	16	60	17
29	56	4	41	30	55	3	42
23	62	10	35	24	61	9	36
48	5	49	28	47	6	50	27
52	25	45	8	51	26	46	7
11	34	22	63	12	33	21	64

Fig. 19.

Magic squares of the 8th order can however be made to combine the pan-diagonal and associated features as shown in Fig. 19 which is contributed by Frierson, and this is true also of all larger squares of order 8*n*.

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